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Dynamics of the classical Heisenberg spin chain†

John A G Roberts and Colin J Thompson

Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract. In this paper we study the evolution equations for the classical discrete anisotropic Heisenberg spin chain. In the continuum limit these equations become completely integrable and are related to well known equations such as the sine-Gordon and non-linear Schrödinger equations. Some particular solutions of the discrete equations are presented, including spin waves, spatially homogeneous solutions and planar states which provide an example of a completely integrable mapping. A linear stability analysis, some numerical studies and particular time-dependent solutions suggest that, for certain regions of phase space and parameter values, the system possesses chaotic solutions.

1. Introduction

There has been considerable interest in recent years in discrete dynamical systems and, in particular, whether the differential-difference equations describing the evolution of such systems are integrable or not (see, for example, Ablowitz and Ladik 1975, 1976). In the so-called continuum limit many of the differential-difference equations studied to date reduce to well known partial differential equations, such as the Korteweg-de Vries equation (KdV), sine-Gordon equation (SG), non-linear Schrödinger equation (NLS) and related equations, which have been shown to be completely integrable by various methods (see, for example, Ablowitz and Segur 1981). Discretised versions of these methods, such as the inverse scattering transform (IST) method, and other more direct methods (Quispel *et al* 1984) have been used to study certain discrete systems but, as in the case of their continuum counterparts, there is, as yet, no systematic method of deciding whether a particular set of partial or differential-difference equations is integrable or not.

Our purpose here is to examine the time evolution of the classical discrete anisotropic Heisenberg spin chain and, in particular, to study the question of integrability of the differential-difference equations describing the evolution of the system.

In the following section we derive the equations of motion for the model and discuss briefly what is known about these equations and their continuum counterpart. In § 3 we present some particular exact solutions of the equations of motion which show that in certain extreme cases the system is integrable. The linearised equations of motion are derived and studied in § 4 and the nature of the stability of these equations

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is examined in some detail. Some numerical and analytical results are given in § 5 which strongly suggest that the model in general is not completely integrable. Our results are summarised in the final section.

2. The equations of motion and the continuum limit

We consider a set of classical three-dimensional vector spins $\boldsymbol{S}_n = (S_{n,1}, S_{n,2}, S_{n,3})$ $n = 1, 2, \ldots, N$, of unit length, with nearest-neighbour interactions only, described by the Hamiltonian

$$\boldsymbol{H} = -\sum_{n=1}^{N-1} (\alpha S_{n,1} S_{n+1,1} + \beta S_{n,2} S_{n+1,2} + \gamma S_{n,3} S_{n+1,3}) = -\sum_{n=1}^{N-1} \boldsymbol{S}_n \cdot \mathbf{J} \boldsymbol{S}_{n+1} \quad (2.1)$$

where the dot in equation (2.1) denotes the ordinary scalar product and

$$\mathbf{J} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \quad (2.2)$$

The time evolution of the system is described by the set of first-order differential equations

$$\mathrm{d}\boldsymbol{S}_n / \mathrm{d}t = \{\boldsymbol{S}_n, \boldsymbol{H}\} \quad (2.3)$$

where the curly brackets in (2.3) denote the Poisson bracket. The standard Poisson bracket for this problem (Mermin 1964) is defined by

$$\{A, B\} = \sum_{n=1}^{N} \sum_{\delta, \mu, \nu = 1}^{3} \varepsilon_{\delta\mu\nu} \frac{\partial A}{\partial S_{n,\delta}} \frac{\partial B}{\partial S_{n,\mu}} S_{n,\nu} \quad (2.4)$$

where $\varepsilon_{\delta\mu\nu}$ is the usual antisymmetric Levi-Civita tensor. The resulting equations of motion are easily shown to be

$$\mathrm{d}\boldsymbol{S}_n / \mathrm{d}t = \boldsymbol{S}_n \times \mathbf{J}(\boldsymbol{S}_{n+1} + \boldsymbol{S}_{n-1}) \quad (2.5)$$

where the cross in (2.5) denotes the ordinary vector product.

The equations of motion given by (2.5) also hold for the corresponding quantum Heisenberg (XYZ) chain where in this case the components of the $\boldsymbol{S}_n$ are operators. The Poisson bracket equation (2.4) is, in fact, designed to agree with the commutation relations for these operators so this result is not very surprising. Alternatively, one could view the classical equations (2.5) as a spin-to-infinity limit of the corresponding quantum equations.

It should also be pointed out that for a finite chain of N spins one should set $\boldsymbol{S}_0 = \boldsymbol{S}_{N+1} = \boldsymbol{0}$ in equation (2.5), whereas for a periodic chain one sets $\boldsymbol{S}_{n+N} \equiv \boldsymbol{S}_n$.

In order to take a continuum limit one introduces a lattice spacing a and a spin field $\boldsymbol{S}(x, t)$ at position x and time t so that when $x = na$

$$\boldsymbol{S}(na, t) \equiv \boldsymbol{S}_n(t). \quad (2.6)$$

Assuming a is small and expanding in a Taylor series we then have, with $x = na$,

$$\boldsymbol{S}_{n+1}(t) + \boldsymbol{S}_{n-1}(t) = 2\boldsymbol{S}(x, t) + a^2 \partial^2 \boldsymbol{S}(x, t)/\partial x^2 + \mathrm{O}(a^4). \quad (2.7)$$

If we then make the replacements

$$\mathbf{J} \to \mathbf{I} + \tfrac{1}{2} a^2 \mathbf{J} \qquad t \to t/a^2 \quad (2.8)$$

in (2.5), where $\mathbf{I}$ denotes the 3×3 unit matrix, and take the (continuum) limit $a\to0$, use of (2.7) results in the partial differential equation

$$\partial \boldsymbol{S}/\partial t = \boldsymbol{S}\times \mathbf{J}\boldsymbol{S} + \boldsymbol{S}\times \partial^2\boldsymbol{S}/\partial x^2 \tag{2.9}$$

for the spin field $\boldsymbol{S}(x, t)$.

Equation (2.9) was first derived phenomenologically by Landau and Lifshitz (1935) and has subsequently been rederived by various people. It formed the basis, for example, for the so-called semiclassical theory of spin waves due to Herring and Kittel (1951).

In the isotropic case $\mathbf{J}=\mathbf{I}$, where the first term in (2.9) vanishes, Lakshmanan *et al* (1976) showed that the spin-wave solution

$$\boldsymbol{S}(x, t) = \{\boldsymbol{a}\cos\theta + \boldsymbol{b}\sin\theta\}\cos\varphi + \boldsymbol{c}\sin\varphi \tag{2.10}$$

where $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a right-handed set of orthogonal unit vectors (see figure 1), φ is constant and

$$\theta(x, t) = px - \omega t \qquad \omega = p^2 \sin\varphi \tag{2.11}$$

is the most general solution of the isotropic equation (2.9), assuming $\boldsymbol{S}(x, t)$ is a function only of the single variable $u = px - \omega t$.

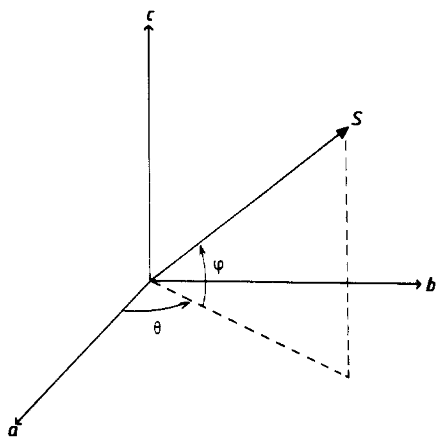

Figure 1. Angular coordinates for spin $\boldsymbol{S}=\boldsymbol{S}(x, t)$ or $\boldsymbol{S}_n(t)$.

They also exhibited a particular solitary wave solution given in the form of equation (2.10) with

$$\sin\varphi(x, t) = [\tanh\tfrac{1}{2}c(x-ct)]^2 \qquad \theta(x, t) = \tan^{-1}[\tanh\tfrac{1}{2}c(x-ct)] + \tfrac{1}{2}cx. \tag{2.12}$$

Various soliton solutions were obtained by Tjon and Wright (1977) and others, and the isotropic continuum equations were subsequently shown by Lakshmanan (1977) and Takhtajan (1977) to be completely integrable, in the sense that there is an infinite number of constants of motion in involution.

Further references can be found in Sklyanin (1979) where it is shown that the anisotropic equation (2.9) is also completely integrable. There it is also shown that in certain limiting cases (2.9) reduces to the SG and NLS equations. These limiting cases are also discussed in Quispel and Capel (1982) where some reductions of the equations of motion for both the discrete chain (2.5) and the continuum chain (2.9) are given.

3. Particular exact solutions of the discrete equations

Here we present some exact analytical solutions of the evolution equations (2.5) for the discrete anisotropic chain.

3.1. Spin-wave solutions

In the isotropic case of (2.5)

$$\mathrm{d}\boldsymbol{S}_n/\mathrm{d}t = \boldsymbol{S}_n \times (\boldsymbol{S}_{n+1} + \boldsymbol{S}_{n-1}) \tag{3.1}$$

it is easily verified that the spin waves

$$\boldsymbol{S}_n(t) = \{\boldsymbol{a}\cos\theta_n + \boldsymbol{b}\sin\theta_n\}\cos\varphi + \boldsymbol{c}\sin\varphi \tag{3.2}$$

form an exact solution when φ is constant and

$$\theta_n(t) = pn - \omega t \qquad \omega = 2(1-\cos p)\sin\varphi. \tag{3.3}$$

It is easy to check that these solutions go over to the continuum spin-wave solutions given by (2.10) and (2.11) by replacing $p \to pa$, $\omega \to \omega/a^2$ (or $t \to t/a^2$) and allowing a to approach zero. Unlike the continuum isotropic case, however, it is almost certain that the above spin-wave solutions of the discrete equations are not the most general solutions of the form $\boldsymbol{S}_n = \boldsymbol{S}(pn - \omega t)$.

In addition it is easily verified that the spin waves given by (3.2) form an exact solution of the anisotropic equations (2.5) for the particular case $\alpha = \beta$, where θ_n is given by equation (3.3) but with $\omega = 2(\gamma - \beta\cos p)\sin\varphi$.

These solutions describe the precessional motion of each spin about the $\boldsymbol{c}$ axis.

3.2. Spatially homogeneous solutions

In the case where

$$\boldsymbol{S}_n(t) = (S_1(t), S_2(t), S_3(t)) \equiv \boldsymbol{S}(t) \tag{3.4}$$

is independent of n, (2.5) reduce to the equations

$$\mathrm{d}S_1/\mathrm{d}t = 2(\gamma-\beta)S_2S_3 \qquad \mathrm{d}S_2/\mathrm{d}t = 2(\alpha-\gamma)S_1S_3 \qquad \mathrm{d}S_3/\mathrm{d}t = 2(\beta-\alpha)S_1S_2. \tag{3.5}$$

Equations (3.5) are equivalent to the equations of motion of a force-free rigid body (Poinsot's motion)—see Cabannes (1968) and Goldstein (1950). The spin $\boldsymbol{S}(t)$ moves on the intersection of the unit sphere defined by the length integral and the quadric surface defined by the energy integral (2.1) with the n dependence removed. The motion is generally a precession about one of the axes with nutation.

Because of the two integrals in S_1, S_2 and S_3, (3.5) can be integrated completely in terms of elliptic functions. The exact solution is given by

$$\begin{aligned} S_1(t) &= \cos\varphi\ \mathrm{cn}(\omega t + \delta) \\ S_2(t) &= (\cos^2\varphi + k^2\sin^2\varphi)^{1/2}\ \mathrm{sn}(\omega t + \delta) \\ S_3(t) &= \sin\varphi\ \mathrm{dn}(\omega t + \delta) \end{aligned} \tag{3.6}$$

where cn, sn and dn are Jacobi elliptic functions, with modulus k, defined by

$$u = \int_0^{\mathrm{sn}\,u} [(1-t^2)(1-k^2t^2)]^{-1/2}\,\mathrm{d}t \tag{3.7}$$

etc, and satisfying

$$\mathrm{cn}^2\, x + \mathrm{sn}^2\, x = \mathrm{dn}^2\, x + k^2\, \mathrm{sn}^2\, x = 1 \tag{3.8}$$

and φ, ω, k and δ are constants.

Direct substitution of (3.6) into (3.5) and use of $(\mathrm{d}/\mathrm{d}x)\,\mathrm{sn}\, x = \mathrm{cn}\, x\, \mathrm{dn}\, x$, etc, shows that (3.5) are satisfied identically provided three equations are satisfied for the three 'unknowns' φ, ω and k with arbitrary phase factor δ. In view of the identities (3.8), it is to be noted that, with the choice of prefactors in (3.6), $\boldsymbol{S}(t)$ has unit length for all φ, ω, k, δ and t.

Unfortunately the above solution does not distinguish discrete from continuous. In other words, (3.5), apart from factors of two, are obtained from the continuum Landau–Lifshitz equation (2.9) by assuming $\boldsymbol{S}(x, t)$ is independent of x. Nevertheless, the elliptic function form of Poinsot's solution leads more or less straightforwardly to exact solutions of the planar form of the equations.

3.3. Planar solutions

If we look for planar solutions of (2.5) with $S_{n,3}(t) \equiv 0$ for all t and n, it is not difficult to show that the only such solutions are stationary solutions, $S_{n,i}(t) = S_{n,i}$, $i = 1, 2$, independent of t, and satisfying

$$\alpha S_{n,2}(S_{n+1,1} + S_{n-1,1}) = \beta S_{n,1}(S_{n+1,2} + S_{n-1,2}) \tag{3.9}$$

with

$$S_{n,1}^2 + S_{n,2}^2 = 1. \tag{3.10}$$

These equations were studied numerically by Thompson *et al* (1985) and the solutions shown to be 'chaotic' in the sense that, for almost all initial conditions, the solutions are aperiodic. 'Chaotic' was an unfortunate choice of word in this case since there is not the 'sensitive dependence on initial conditions' required of truly chaotic, as opposed to simply aperiodic, behaviour. Moreover, as we will see in a moment, it is not difficult to show that solutions of (3.9) and (3.10) lie on continuous invariant curves in the $(S_{n+1,1}, S_{n,1})$ phase plane.

Thus if we multiply (3.9) on both sides by $(S_{n+1,1} - S_{n-1,1})$ and use (3.10) it easily follows that either

$$S_{n+1,2} = -S_{n-1,2} \tag{3.11}$$

or

$$\alpha S_{n,2}(S_{n-1,2} - S_{n+1,2}) = \beta S_{n,1}(S_{n+1,1} - S_{n-1,1})$$

which, on rearranging and iterating, implies that

$$S_{n,1}S_{n+1,1} + \lambda S_{n,2}S_{n+1,2} = C \tag{3.12}$$

where $\lambda = \alpha/\beta$ and C is a constant.

More generally, it is not difficult to show that arbitrary stationary states of (2.5), i.e. solutions satisfying

$$\boldsymbol{S}_n \times \mathbf{J}(\boldsymbol{S}_{n+1} + \boldsymbol{S}_{n-1}) = \mathbf{0} \tag{3.13}$$

admit constants of the motion of the form (3.12). Thus if we take the vector product on both sides of (3.13) with $\mathbf{J}^{-1}(\boldsymbol{S}_{n+1} - \boldsymbol{S}_{n-1})$ it easily follows that either $\boldsymbol{S}_{n+1} = -\boldsymbol{S}_{n-1}$, corresponding to a flip of every second spin, or

$$\mathbf{J}^{-1}\boldsymbol{S}_{n+1} \cdot \boldsymbol{S}_n = \text{constant}. \tag{3.14}$$

The set of constants (3.12) for planar states was found independently by Granovskii and Zhedanov (1986) and more general methods for integrating stationary states of discrete dynamical systems have been given recently by Quispel *et al* (1988a, b).

As mentioned above, equations (3.9) for planar states and their invariant curves (3.12) can be integrated completely by an elliptic substitution. The results, also found independently by Granovskii and Zhedanov (1986), are as follows.

By symmetry we can assume that the anisotropy parameter $\lambda = \alpha/\beta$ is in the interval (0, 1). For given $0 < \lambda < 1$ and initial conditions such that C in (3.12) satisfies $\lambda > C$, the solution of (3.9) is given by

$$S_{n,1} = \operatorname{cn}(pn + \delta) \qquad S_{n,2} = \operatorname{sn}(pn + \delta) \tag{3.15}$$

where δ is arbitrary, p is such that

$$\operatorname{dn} p = \lambda \tag{3.16}$$

and the modulus k of the elliptic functions is given by

$$k^2 = (1 - \lambda^2)(1 - C^2)^{-1}. \tag{3.17}$$

On the other hand, when $\lambda < C < 1$ we have the solutions

$$S_{n,1} = \operatorname{dn}(pn + \delta) \qquad S_{n,2} = k \operatorname{sn}(pn + \delta) \tag{3.18}$$

where p is such that

$$\operatorname{cn} p = \lambda \tag{3.19}$$

and the modulus is given by

$$k^2 = (1 - C^2)(1 - \lambda^2)^{-1}. \tag{3.20}$$

The above solutions cover all possible choices of initial conditions and moreover the invariant curves (3.12) fill the phase space $|S_{n,1}| \leqslant 1, |S_{n+1,1}| \leqslant 1$ densely. In this sense (3.9) for planar states are completely integrable.

Notice that since $\operatorname{sn} x$ and $\operatorname{cn} x$ have period $4K(k)$ where $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} \, d\theta \tag{3.21}$$

we have periodic (point) solutions or cycles when there exist relatively prime integers n_1 and n_2 such that

$$p = 4K(k) n_1 / n_2. \tag{3.22}$$

In other words, from (3.16) and (3.19), for given λ we have cycles of length $N(\lambda)$ when

$$\lambda = \operatorname{dn}[4K(k)/N(\lambda)] \qquad \text{or} \qquad \lambda = \operatorname{cn}[4K(k)/N(\lambda)] \tag{3.23}$$

in accordance with the numerical solutions of Thompson *et al* (1985).

4. Linear stability analysis

As observed by Thompson *et al* (1985) the behaviour of the planar solutions can be understood quite well by studying the corresponding linearised equations. Here we investigate the linearised stability of (2.5) from initial states that are close to planar.

For convenience we choose a polar representation for the spins in terms of angles $\theta_n(t)$ and $\varphi_n(t)$ shown in figure 1, i.e.

$$S_n(t)=\{\boldsymbol{a}\cos\theta_n+\boldsymbol{b}\sin\theta_n\}\cos\varphi_n+\boldsymbol{c}\sin\varphi_n. \tag{4.1}$$

The set of non-linear differential-difference equations (2.5) in terms of θ_n and φ_n is given by

$$\begin{aligned}\mathrm{d}\theta_n/\mathrm{d}t=\alpha(\cos\theta_{n-1}\cos\varphi_{n-1}+\cos\theta_{n+1}\cos\varphi_{n+1})\tan\varphi_n\cos\theta_n\\ +\beta(\sin\theta_{n-1}\cos\varphi_{n-1}+\sin\theta_{n+1}\cos\varphi_{n+1})\tan\varphi_n\sin\theta_n\\ -\gamma(\sin\varphi_{n-1}+\sin\varphi_{n+1})\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\mathrm{d}\varphi_n/\mathrm{d}t=-\alpha(\cos\theta_{n-1}\cos\varphi_{n-1}+\cos\theta_{n+1}\cos\varphi_{n+1})\sin\theta_n\\ +\beta(\sin\theta_{n-1}\cos\varphi_{n-1}+\sin\theta_{n+1}\cos\varphi_{n+1})\cos\theta_n.\end{aligned} \tag{4.3}$$

The planar solutions studied in the previous section correspond to $\varphi_n=0$ in (4.2) and (4.3). Here we consider the above equations linearised about the trivial stationary solution $\theta_n=\varphi_n=0$. Thus, if we expand the right-hand sides of (4.2) and (4.3) about this point, and retain only linear terms, we arrive at the simple linearised equations

$$\mathrm{d}\theta_n/\mathrm{d}t=-\gamma\varphi_{n-1}+2\alpha\varphi_n-\gamma\varphi_{n+1} \qquad \mathrm{d}\varphi_n/\mathrm{d}t=\beta\theta_{n-1}-2\alpha\theta_n+\beta\theta_{n+1}. \tag{4.4}$$

The linearised planar equations, with $\varphi_n=0$, are

$$\theta_{n-1}-2\lambda\theta_n+\theta_{n+1}=0 \qquad (\lambda=\alpha/\beta) \tag{4.5}$$

with particular solutions

$$\theta_n=\theta_1\cos(n-1)\alpha \qquad \text{when} \qquad \lambda=\cos\alpha \tag{4.6}$$

matching extremely well with the exact planar solutions discussed in the previous section for small initial θ_1 (and θ_2).

The linearised time-dependent equations (4.4) can be expressed in matrix form:

$$\mathrm{d}\boldsymbol{x}_n/\mathrm{d}t=\mathbf{b}\boldsymbol{x}_{n-1}+\mathbf{a}\boldsymbol{x}_n+\mathbf{b}\boldsymbol{x}_{n+1} \tag{4.7}$$

where

$$\boldsymbol{x}_n=\begin{pmatrix}\theta_n\\ \varphi_n\end{pmatrix} \qquad \mathbf{a}=\begin{pmatrix}0 & 2\alpha\\ -2\alpha & 0\end{pmatrix} \qquad \mathbf{b}=\begin{pmatrix}0 & -\gamma\\ \beta & 0\end{pmatrix}. \tag{4.8}$$

If we then assume, for simplicity, that we have a periodic chain of N spins ($\boldsymbol{x}_{n+N}\equiv\boldsymbol{x}_n$), (4.7) can be expressed in circulant $N\times N$ matrix form with 2×2 entries $\mathbf{a}$ on the main diagonal and entries $\mathbf{b}$ on the two nearest-neighbour diagonals.

In this case the solution of (4.7) is easily found to be

$$\boldsymbol{x}_n(t)=\sum_{k=1}^{N}\exp(\mathbf{D}_k t)N^{-1}\sum_{p=1}^{N}\boldsymbol{x}_p(0)\cos(2\pi k/N)(p-n) \tag{4.9}$$

where

$$\mathbf{D}_k=\begin{pmatrix}0 & 2U_k\\ -2V_k & 0\end{pmatrix} \tag{4.10}$$

and

$$U_k=\alpha-\gamma\cos(2\pi k/N) \qquad V_k=\alpha-\beta\cos(2\pi k/N). \tag{4.11}$$

The important point to recognise here is that the linearised equations are (locally) unstable if one of the eigenvalues $\pm 2(-U_k V_k)^{1/2}$ of $\mathbf{D}_k$ for any k is positive, i.e. if U_k and V_k have opposite sign for any k. The linearised equations are also unstable if just one of U_k or V_k is zero for any k, leading to a repeated 0 eigenvalue. Otherwise the eigenvalues of $\mathbf{D}_k$ are pure imaginary for all k and the system is neutrally stable, i.e. the $x_n(t)$ given by (4.9) are linear combinations of (bounded) oscillatory solutions of (4.7).

The situation is summarised in the cross section of the (α, β, γ) parameter space in figure 2, i.e. for fixed α, the linearised system is neutrally stable in the square $|\gamma| < |\alpha|$ and $|\beta| < |\alpha|$ and on the line $\gamma = \beta$ for all chain lengths N, and on the square boundaries $\gamma = -\alpha$ and $\beta = -\alpha$ for odd values of N. For all other parameter values the linearised system is unstable for most N. In particular, in the exterior of the square, excluding the line $\gamma = \beta$, we can definitely say that the zero solution of the non-linear system is also unstable for most N. The neighbourhood of the origin in these systems thus becomes a prime candidate for possible chaotic behaviour.

Some numerical and analytic results given in the following section support the claim that the discrete Heisenberg chain does, in fact, have chaotic solutions in some regions of parameter and phase space.

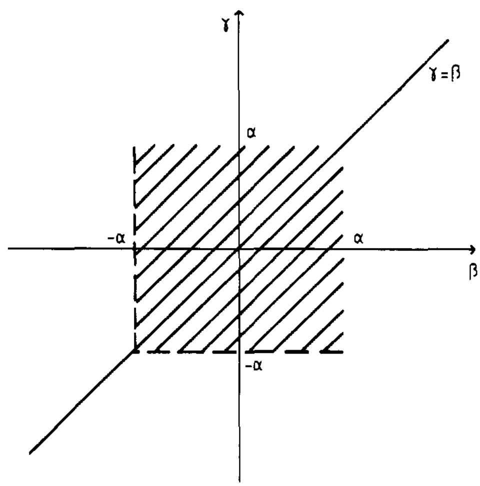

Figure 2. Shaded area and the full and broken lines indicating the region of linearised stability for the case where α is positive.

5. Regular and chaotic solutions of the non-linear problem

It is clear from the analysis of the previous section that, unless one excites one or a small number of periodic neutrally stable modes of the linearised system, it will be virtually impossible to distinguish numerically between regular and chaotic behaviour of the corresponding non-linear system, particularly if the periods of the underlying locally stable modes are incommensurate. This is certainly borne out by numerical studies of the non-linear equations for periodic chains consisting of small numbers of spins. Some typical results are shown in figure 3 for parameter values where the linearised equations are neutrally stable and locally unstable.

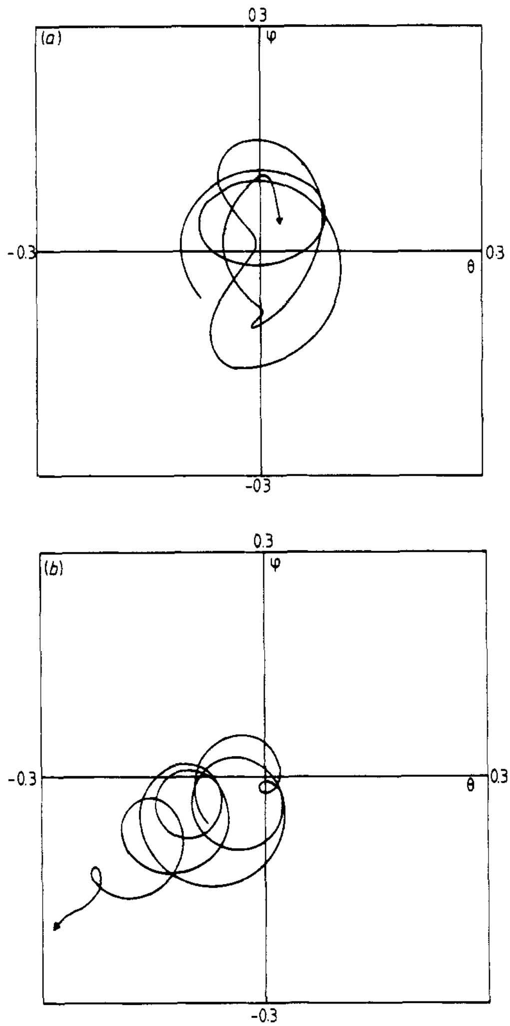

Figure 3. Numerical solutions of non-linear equations (4.2) and (4.3) for a short chain ($N = 17$). Shown is the motion of a typical spin in the two cases where the linearised system is (a) stable ($\alpha = 1.0$, $\beta = 0.3$, $\gamma = -0.87$) and (b) unstable ($\alpha = 1.0$, $\beta = -1.1$, $\gamma = -0.87$). The initial condition in both cases is the same and close to the zero stationary solution.

The best evidence we have for chaotic behaviour of the non-linear discrete system is the spatial chaos found from the study of the non-linear equations (4.2) and (4.3) for the case $\alpha = \beta$ and $\theta_n = \theta(t)$, independent of n. In this case (4.3) implies $\mathrm{d}\varphi_n/\mathrm{d}t = 0$ and equation (4.2) reduces to

$$(\mathrm{d}\theta/\mathrm{d}t) \cos \varphi_n = \alpha(\cos \varphi_{n-1} + \cos \varphi_{n+1}) \sin \varphi_n - \gamma(\sin \varphi_{n-1} + \sin \varphi_{n+1}) \cos \varphi_n. \tag{5.1}$$

It follows that we must have $\theta(t) = \omega t + \delta$ and the condition

$$\omega \cos \varphi_n = \alpha(\cos \varphi_{n-1} + \cos \varphi_{n+1}) \sin \varphi_n - \gamma(\sin \varphi_{n-1} + \sin \varphi_{n+1}) \cos \varphi_n. \tag{5.2}$$

Equation (5.2) can be regarded as a mapping in the $(\varphi_n, \varphi_{n+1})$ plane. It can be recast in the symmetric form

$$\sin(\varphi_{n+1} - f(\varphi_n)) = -\sin(\varphi_{n-1} - f(\varphi_n)) - \omega g(\varphi_n) \tag{5.3}$$

where

$$\cos f(\varphi_n) = \gamma \cos \varphi_n / \rho(\varphi_n) \qquad \sin f(\varphi_n) = \alpha \sin \varphi_n / \rho(\varphi_n)$$
$$g(\varphi_n) = \cos \varphi_n / \rho(\varphi_n) \tag{5.4}$$

and

$$\rho(\varphi_n) = (\alpha^2 \sin^2 \varphi_n + \gamma^2 \cos^2 \varphi_n)^{1/2}. \tag{5.5}$$

Mappings of the form (5.3) and their properties (with a different $g(\varphi_n)$) were discussed in Belobrov *et al* (1984) in connection with the equilibrium structures of a planar Heisenberg chain in the presence of an external field.

When $\omega = 0$ the mapping (5.3)-(5.5) reduces to

$$\varphi_{n+1} + \varphi_{n-1} = 2 \tan^{-1}(K \tan \varphi_n) \tag{5.6}$$

where $K = \alpha/\gamma$. This is the angular form of the integrable mapping (3.9) and (3.10) which has only periodic or quasiperiodic behaviour.

When $\omega \neq 0$ then the whole phase space $-\pi < \varphi_{n-1} \leqslant \pi$, $-\pi < \varphi_n \leqslant \pi$ is not available to the mapping (5.3) on account of the existence of so-called forbidden regions. These regions comprise those points $(\varphi_{n-1}, \varphi_n)$ which make the right-hand side of (5.3) fall outside the range $[-1, 1]$ and so make the calculation of the next point impossible. Trajectories with initial conditions (φ_0, φ_1) which escape at some iteration to a forbidden region become invalid.

For the case $\omega \neq 0$ and for the isotropic chain $\alpha = \beta = \gamma$, equations (5.3)-(5.5) become

$$\sin(\varphi_{n+1} - \varphi_n) = \sin(\varphi_n - \varphi_{n-1}) - (\omega/\alpha) \cos \varphi_n \tag{5.7}$$

which has been studied by Slot (1982) and, more recently, by Ananthakrishna *et al* (1987) where it was shown to have chaotic trajectories for certain initial spin values (φ_0, φ_1) and a range of the parameter $|\omega/\alpha|$.

In the most general case of (5.3)-(5.5) with $\omega \neq 0$ and $\alpha \neq \gamma$, we also find chaotic trajectories present in the $(\varphi_n, \varphi_{n+1})$ plane for some values of α, γ and ω and for some initial conditions. A typical example is shown in figures 4 and 5. These solutions correspond to solutions of the uniaxial $(\alpha = \beta)$ chain which are periodic in t (with period $2\pi/\omega$) but the spin component amplitudes exhibit chaotic behaviour.

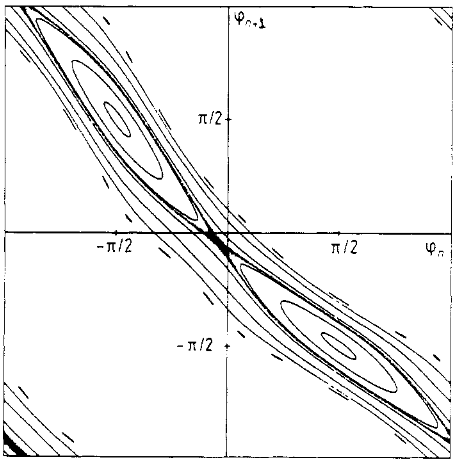

Figure 4. Phase portrait of the mapping (5.3)-(5.5) for $\alpha = \beta = -1.05$, $\gamma = 1.0$ and $\omega = 0.55$. The points $(-\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$ are a 2-cycle of the mapping.

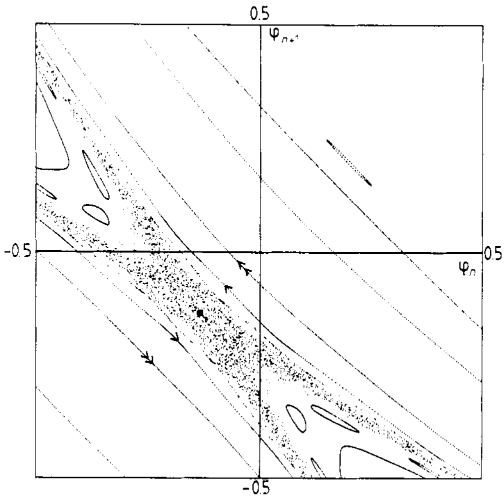

Figure 5. Enlargement of the area around the origin in figure 4 (note that the islands shown here are not drawn in figure 4). The bold dot is the hyperbolic fixed point at $\varphi_0 = \varphi_1 \approx -0.1346$. The single and double arrows indicate parts of two different curve trajectories. The chaotic trajectory is produced by iterating the point $\varphi_0 = \varphi_1 = -0.15$.

The fact that some solutions of our equations exhibit chaotic behaviour does not, of course, rule out the possibility that in some other regions of phase space and parameter values our system is completely integrable.

6. Conclusions

In this paper we studied the evolution equations for the classical discrete anisotropic Heisenberg spin chain. We reviewed the situation in the continuum limit where the equations become completely integrable and in special limiting cases reduce to the well known sine-Gordon and non-linear Schrödinger equations.

Some particular solutions of the discrete evolution equations were discussed including spin-wave solutions, spatially homogeneous solutions which reproduce Poinsot's motion and planar solutions where the equations afford an example of a completely integrable mapping.

A linear stability analysis of the discrete equations was performed which indicates the possibility of irregular or chaotic solutions. This was borne out by some numerical studies and some particular solutions which are periodic in time but with numerical indications of chaotic spin component amplitudes in small regions of phase space and parameter values.

The question of integrability of the time-dependent non-planar equations in some region of phase space remains an open question.

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References

Ablowitz M J and Ladik J F 1975 *J. Math. Phys.* **16** 598
—— 1976 *J. Math. Phys.* **17** 1011
Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
Ananthakrishna G, Balakrishnan R and Hao B-L 1987 *Phys. Lett.* **121A** 407
Belobrov P I, Beloshapkin V V, Zaslavsky G M and Tret'yakov A G 1984 *Sov. Phys.-JETP* **60** 180
Cabannes H 1968 *General Mechanics* (Waltham, MA: Blaisdell) 2nd edn, p 264
Goldstein H 1950 *Classical Mechanics* (Reading, MA: Addison-Wesley) p 159
Granovskii Ya I and Zhedanov A S 1986 *JETP Lett.* **44** 304
Herring C and Kittel C 1951 *Phys. Rev.* **81** 869
Lakshmanan M 1977 *Phys. Lett.* **61A** 53
Lakshmanan M, Ruijgrok Th W and Thompson C J 1976 *Physica* **84A** 577
Landau L D and Lifshitz E M 1935 *Phys. Z. Sowj.* **8** 15
Mermin N D 1964 *Phys. Rev.* **134** A112
Quispel G R W and Capel H W 1982 *Physica* **110A** 41
Quispel G R W, Nijhoff F W, Capel H W and Van der Linden J 1984 *Physica* **125A** 344
Quispel G R W, Roberts J A G and Thompson C J 1988a *Phys. Lett.* **126A** 419
—— 1988b *Integrable Mappings and Soliton Equations II* to be published
Sklyanin E K 1979 *LOMI preprint* E-3
Slot J J M 1982 *Master's Thesis* Department of Theoretical Physics, Utrecht
Takhtajan L A 1977 *Phys. Lett.* **64A** 235
Thompson C J, Ross K A, Thompson B J P and Lakshmanan M 1985 *Physica* **133A** 330
Tjon J and Wright J 1977 *Phys. Rev.* B **15** 3470